Izračunati $I=\int_0^2 x\sqrt{4+x^2}\,\operatorname{arctg}\frac{x}{2}\,dx$.

Rj. $\int_0^2 x\sqrt{4+x^2}\,\operatorname{arctg}\frac{x}{2}\,dx = \left|\begin{array}{ll} u=\operatorname{arctg}\frac{x}{2} & dv=x\sqrt{4+x^2} \\ du=\frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2} & v\overset{(*)}{=}\frac{1}{3}(4+x^2)^{\frac{3}{2}} \end{array}\right| \overset{(\square)}{=}$

$$\int x\sqrt{4+x^2}\,dx = \left|\begin{array}{l} 4+x^2=s^2 \\ 2x\,dx=2s\,ds \\ x\,dx=s\,ds \end{array}\right| = \int s^2\,ds = \frac{1}{3}s^3+C=\frac{1}{3}(4+x^2)^{\frac{3}{2}}+C \quad \ldots(*)$$

$$\overset{(\square)}{=} \frac{1}{3}(4+x^2)^{\frac{3}{2}}\operatorname{arctg}\frac{x}{2}\Big|_0^2-\frac{1}{3}\cdot\frac{1}{2}\int_0^2\frac{1\cdot 4}{\left(1+\frac{x^2}{4}\right)\cdot 4}(4+x^2)^{\frac{3}{2}}\,dx=$$

$$=\frac{1}{3}\left[\sqrt{8^3}\operatorname{arctg}1-0\right]-\frac{1}{6}\cdot 4\int_0^2\frac{(4+x^2)^{\frac{3}{2}}}{4+x^2}\,dx=$$

$$=\frac{1}{3}\cdot 8\cdot 2\sqrt{2}\cdot\frac{\pi}{4}-\frac{2}{3}\int_0^2\sqrt{4+x^2}\,dx=\frac{4}{3}\pi\sqrt{2}-\frac{2}{3}\int_0^2\sqrt{4+x^2}\,dx \quad \ldots(1)$$

Trebamo još izračunati $\int_0^2\sqrt{4+x^2}\,dx$:

$$J=\int_0^2\sqrt{4+x^2}\,dx=\left|\begin{array}{ll} u=\sqrt{4+x^2} & dv=dx \\ du=\frac{x}{\sqrt{4+x^2}}dx & v=x \end{array}\right|=x\sqrt{4+x^2}\Big|_0^2-\int_0^2\frac{x^2+4-4}{\sqrt{4+x^2}}\,dx$$

$$=2\sqrt{8}-\int_0^2\sqrt{4+x^2}\,dx+4\int_0^2\frac{dx}{\sqrt{x^2+4}}=4\sqrt{2}-J+4\ln\left|x+\sqrt{x^2+4}\right|\Big|_0^2=$$

$$=4\sqrt{2}-J+4\left(\ln(2+2\sqrt{2})-\ln 2\right)=4\sqrt{2}-J+4\ln(1+\sqrt{2}) \implies$$

$$2J=4\sqrt{2}+4\ln(1+\sqrt{2}) \implies J=\int_0^2\sqrt{4+x^2}\,dx=2\sqrt{2}+2\ln(1+\sqrt{2}). \quad \ldots(2)$$

$$I=\int_0^2 x\sqrt{4+x^2}\,\operatorname{arctg}\frac{x}{2}\,dx \overset{(1)\,i\,(2)}{=} \frac{4}{3}\pi\sqrt{2}-\frac{4}{3}\sqrt{2}-\frac{4}{3}\ln(1+\sqrt{2})$$

$$=\frac{4}{3}\left[\sqrt{2}(\pi-1)-\ln(1+\sqrt{2})\right]$$

traženo rješenje

Promjeniti poredak integracije u integralu

$$I=\int_{-7}^{1}dy\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x,y)\,dx$$

Rj.

Ako sa D označimo oblast integracije imamo

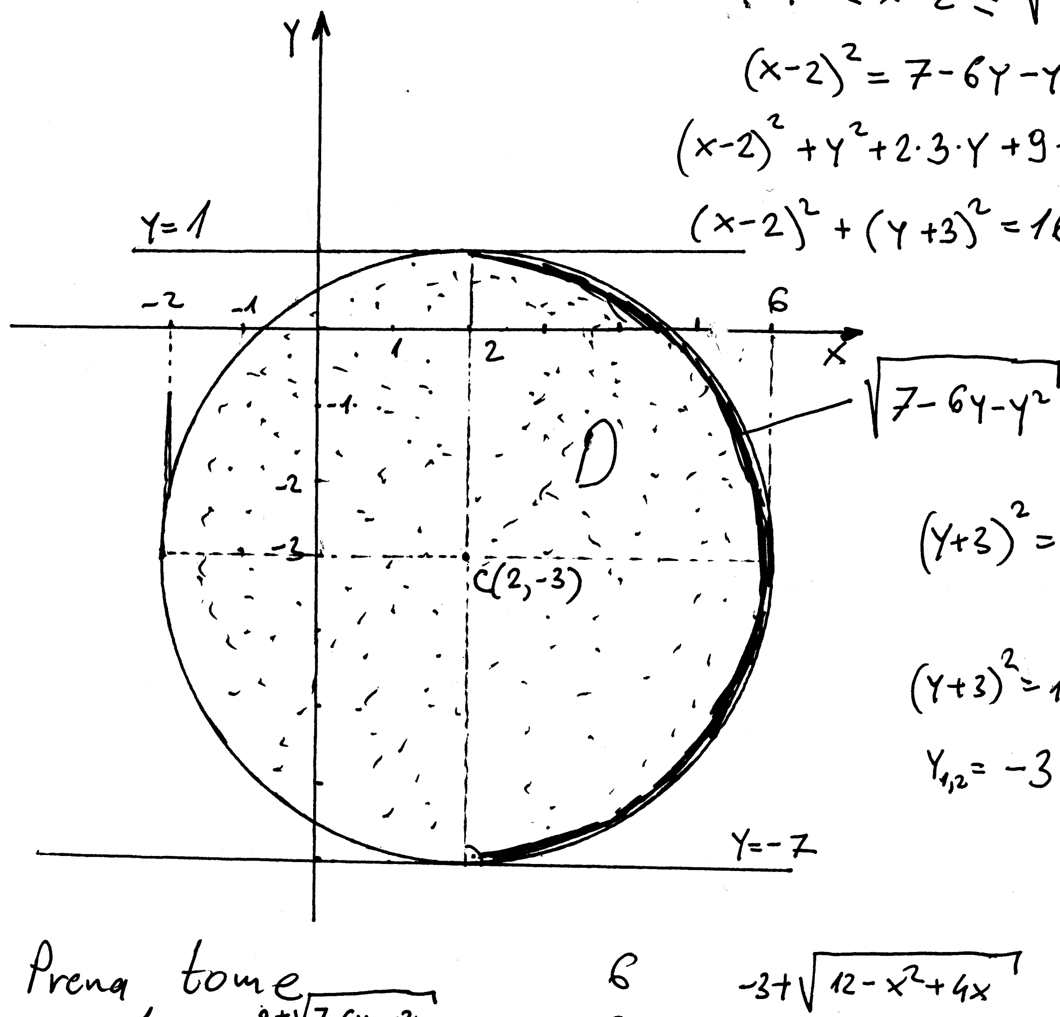
$$D:\begin{cases} 2-\sqrt{7-6y-y^2}\le x\le 2+\sqrt{7-6y-y^2} \\ -7\le y\le 1. \end{cases}$$

Posmatrajmo prvo samo promjenjivu x: $\quad -\sqrt{7-6y-y^2}\le x-2\le\sqrt{7-6y-y^2}$

$$(x-2)^2=7-6y-y^2$$

$$(x-2)^2+y^2+2\cdot 3\cdot y+9-9=7$$

$$(x-2)^2+(y+3)^2=16$$ krug sa centrom u tački $C(2,-3)$ poluprečnika $r=4$



$$(y+3)^2=16-\underbrace{(x-2)^2}_{x^2-4x+4}$$

$$(y+3)^2=12-x^2+4x$$

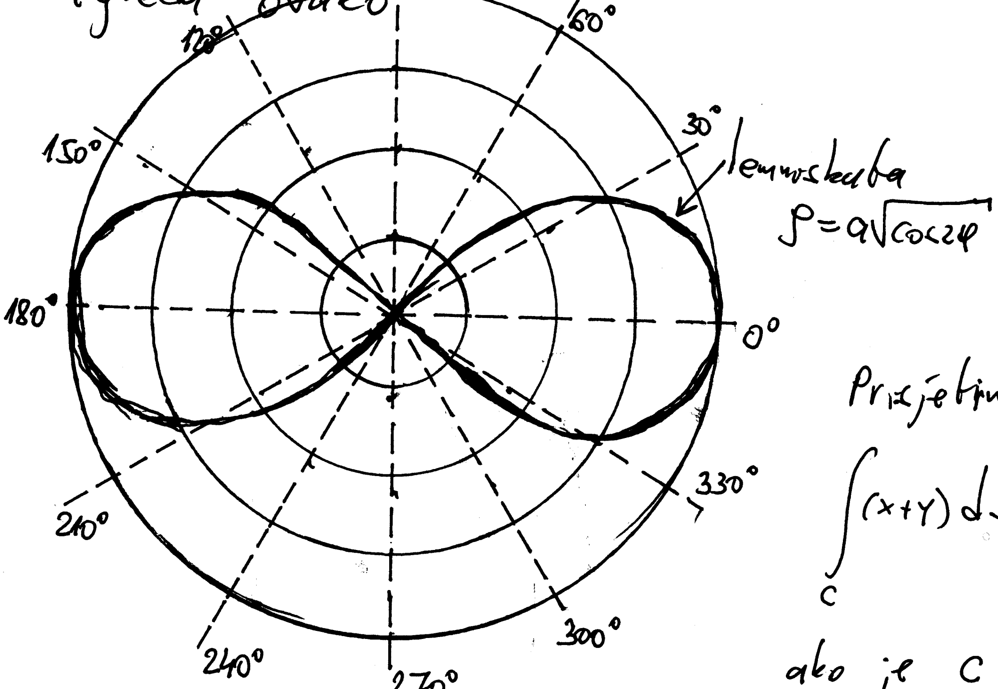
$$y_{1,2}=-3\pm\sqrt{12-x^2+4x}$$

Prema tome

$$I=\int_{-7}^{1}dy\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x,y)\,dx=\int_{-2}^{6}dx\int_{-3-\sqrt{12-x^2+4x}}^{-3+\sqrt{12-x^2+4x}} f(x,y)\,dx.$$

(#) Izračunati krivoliniski integral prve vrste $\int_c (x+y)\,dS$, ako je c desna latica lemniskate $\rho = a\sqrt{\cos 2\varphi}$.

Rj: Lemniskata $\rho = a\sqrt{\cos 2\varphi}$ u polarnom koordinatnom sistemu izgleda ovako



Data kriva je prikazana u polarnim koordinatama

$$c:\begin{cases}\rho = a\sqrt{\cos 2\varphi}\\ \varphi \in \left[-\frac{\pi}{4},\frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4},\frac{5\pi}{4}\right]\end{cases}$$

Prisjetimo se,

$$\int_c (x+y)\,dS = \int_{t_1}^{t_2} (\eta(t)+\mu(t))\sqrt{(\eta'(t))^2+(\mu'(t))^2}\,dt$$

ako je c data u obliku

$$c:\begin{cases}x=\eta(t)\\ y=\mu(t)\\ t_1 \le t \le t_2\end{cases}.$$

$\int_{c'} (x+y)\,dS = \Bigg|$ kao pomoć uvedimo polarne koordinate $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Za ρ ćemo uzeti $\rho = a\sqrt{\cos 2\varphi}$. Prema tome

desna latica lemniskate

$$c':\begin{cases}x = a\cos\varphi\sqrt{\cos 2\varphi}\\ y = a\sin\varphi\sqrt{\cos 2\varphi}\\ -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}\end{cases}$$

zato što posmatramo desnu stranu lemniskate

$$x' = \left(a(-\sin\varphi)\sqrt{\cos 2\varphi} + a\cos\varphi\cdot\frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}}\cdot(-\sin 2\varphi)\cdot 2\right)d\varphi = a\left(-\sin\varphi\sqrt{\cos 2\varphi} - \cos\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right)d\varphi \overset{\text{adic. teor.}}{=} -a\frac{\sin 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$y' = \left(a\cos\varphi\sqrt{\cos 2\varphi} + a\sin\varphi\cdot\frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}}(-\sin 2\varphi)\cdot 2\right)d\varphi = a\cos\varphi\sqrt{\cos 2\varphi} - a\sin\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}} \overset{\text{adiciona teor.}}{=} \dots = a\frac{\cos 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$x'^2 + y'^2 = a^2\frac{\sin^2 3\varphi}{\cos 2\varphi}d\varphi^2 + a^2\frac{\cos^2 3\varphi}{\cos 2\varphi}d\varphi^2 = a^2\frac{1}{\cos 2\varphi}d\varphi^2 \Bigg| =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cancel{\sqrt{\cos 2\varphi}}\; a(\cos\varphi + \sin\varphi)\cdot a\frac{1}{\cancel{\sqrt{\cos 2\varphi}}}\,d\varphi = a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\varphi + \sin\varphi)\,d\varphi =$$

$$= a^2\Big(\overbrace{\sin\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}^{\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}\right)} - \overbrace{\cos\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}^{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}\Big) = a^2\sqrt{2}$$ traženo rješenje.

(#) Odrediti brojeve a i b tako da vektorsko polje $\vec{v}=(yz+axy,\ xz+bx^2+yz^2,\ axy+y^2z)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravoliniske konture od tačke $A(1,1,1)$ prema tački $B(2,2,2)$.

Rj: Za vektorsko polje $\underbrace{\vec{v}}_{(\vec{v}=(v_x,v_y,v_z))}$ kažemo da je potencijalno ako je $rot\,\vec{v}=\vec{0}$. Znamo da

$$rot\,\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}$$

$$rot\,\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz+axy & xz+bx^2+yz^2 & axy+y^2z\end{vmatrix}=$$

$$=\left(ax+\cancel{2yz}-x-\cancel{2yz},\ -(ay-y),\ z+2bx-z-ax\right)$$

$$=(ax-x,\ y-ay,\ 2bx-ax)$$

$$rot\,\vec{v}=\vec{0}\quad\Rightarrow\quad\begin{array}{l}ax-x=0\\ y-ay=0\\ 2bx-ax=0\end{array}\qquad\begin{array}{l}a=1\\ b=\frac{1}{2}\end{array}$$

Za $a=1$ i $b=\frac{1}{2}$ vektorsko polje $\vec{v}$ je potencijalno polje.

Cirkulaciju vektorskog polja $\vec{v}$ duž krive c tražimo po formuli

$$C=\int_c \vec{v}\,d\vec{r}=\int_c v_x\,dx+v_y\,dy+v_z\,dz$$

Kriva c je dio prave od tačke $A(1,1,1)$ $_{(x_1,y_1,z_1)}$ do tačke $B(2,2,2)$ $^{(x_2,y_2,z_2)}$.

Kako glasi jednačina prave u prostoru kroz dvije tačke?

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\quad\Rightarrow\quad\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{1}\quad(=t)\qquad\begin{array}{l}x-1=t\\ y-1=t\\ z-1=t\end{array}$$

Krivca c u parametarskom obliku

$$c: \begin{cases} x = t+1 \\ y = t+1 \\ z = t+1 \\ 0 \le t \le 1 \end{cases} \qquad \begin{aligned} dx &= dt \\ dy &= dt \\ dz &= dt \end{aligned}$$

U našem slučaju

$$C = \int_c (yz + xy)\,dx + \left(xz + \frac{1}{2}x^2 + yz^2\right)dy + (xy + y^2 z)\,dz =$$

$$= \int_0^1 \left[(t+1)^2 + (t+1)^2 + (t+1)^2 + \frac{1}{2}(t+1)^2 + (t+1)^3 + (t+1)^2 + (t+1)^3\right] dt =$$

$$= \left| \, d(t+1) = dt \, \right| = \int_0^1 \left[\frac{9}{2}(t+1)^2 + 2(t+1)^3\right] d(t+1) =$$

$$= \frac{9}{2} \cdot \frac{(t+1)^3}{3}\Big|_0^1 + 2 \cdot \frac{(t+1)^4}{4}\Big|_0^1 = \frac{9}{6}(8-1) + \frac{1}{2}(16-1)$$

$$= \frac{63}{6} + \frac{15 \cdot 3}{2 \cdot 3} = \frac{108}{6} = 18$$

traženo rješenje